Optimal Trajectories in Discrete Space with Acceleration Constraints

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ALGO Seminar

February 27th, 2026

Outline

Quick background

VectorRacer a discrete model embedding physics of movements in a clever way[1].

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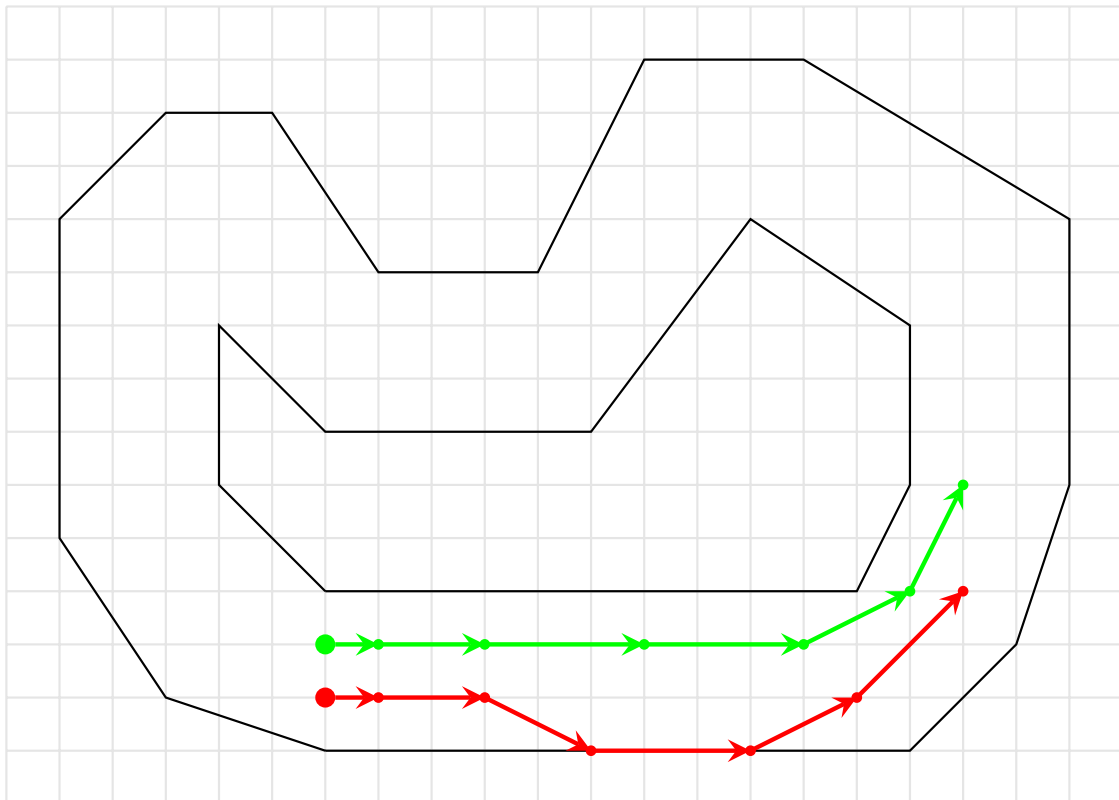
Originally, a paper-and-pencil game among two players in a *racetrack*:



Motivation and contributions

Motion planning in an open space without track limits (for example, drones delivering packages)

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We revisit the techniques of Bekos et al. [2]:

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We revisit the techniques of Bekos et al. [2]:

$\rightarrow$ Improve and extend the constant time procedure to recover a succint representation of all the *feasible branching times* among two configurations in *any* dimension d;

Furthermore,

- We provide a constant time (and space) strategy to characterize an explicit trajectory among two configurations;
- We prove that the speed in an open space setting is not trivially bounded;

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A d-dimensional generalization of Gardner's model in $\mathbb{Z}^d$.

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A *trajectory* is a sequence of configurations $c_0, \ldots, c_\ell$ where every two consecutive configurations satisfy the above conditions.

Branching Cost

Problem: Branching Cost

In: Two configurations $c = (p, v)$ and $c' = (p', v')$ in $\mathbb{Z}^d \times \mathbb{Z}^d$

Out: What is the shortest length of a trajectory from c to c'?

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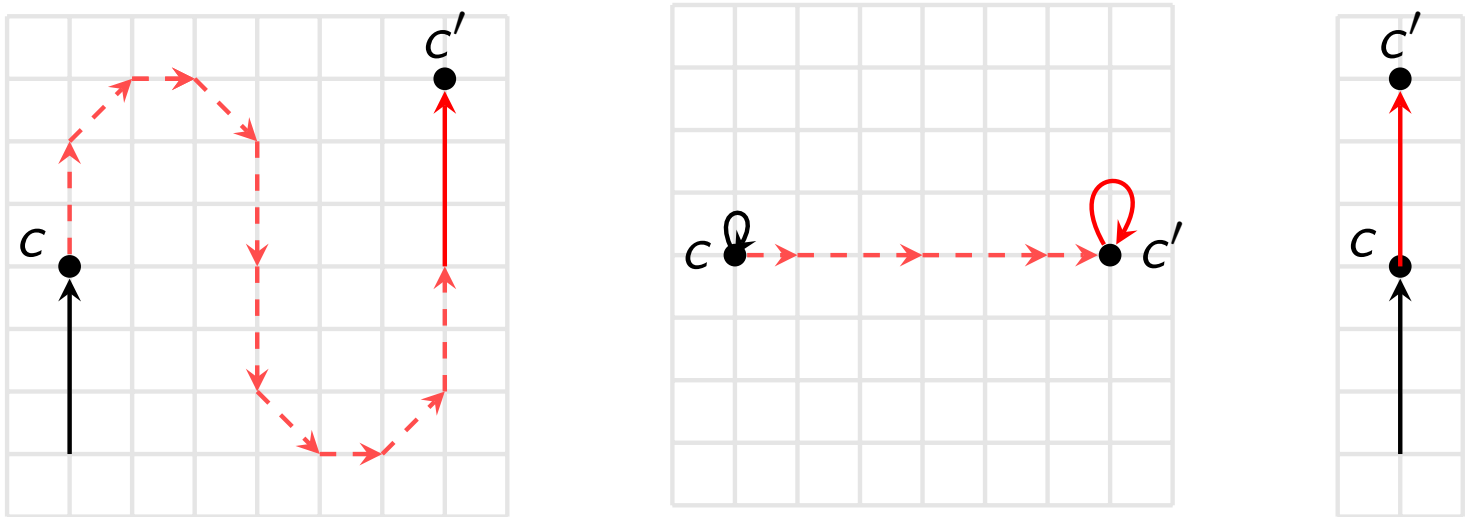

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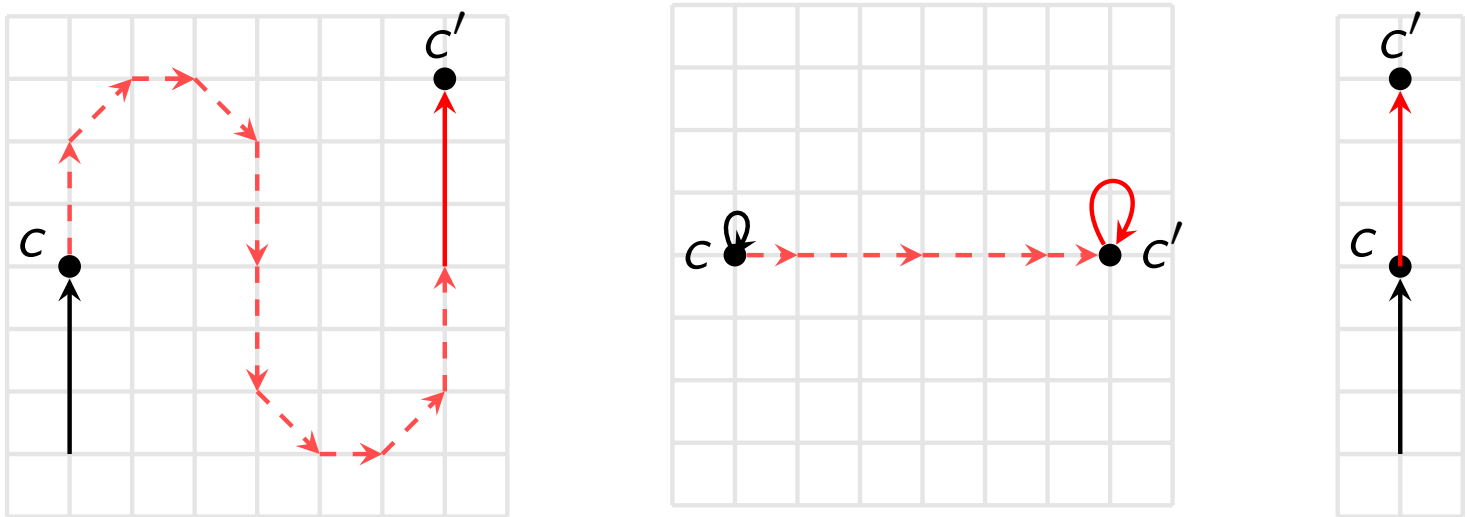

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- However, the minimum time in x is 5 timesteps, and 1 timestep for y

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Given $c = (x, s)$ and $c' = (x', s')$ (one-dimensional), $\mathcal{I}(c, c')$ returns the set of feasible time for branching c to c'.

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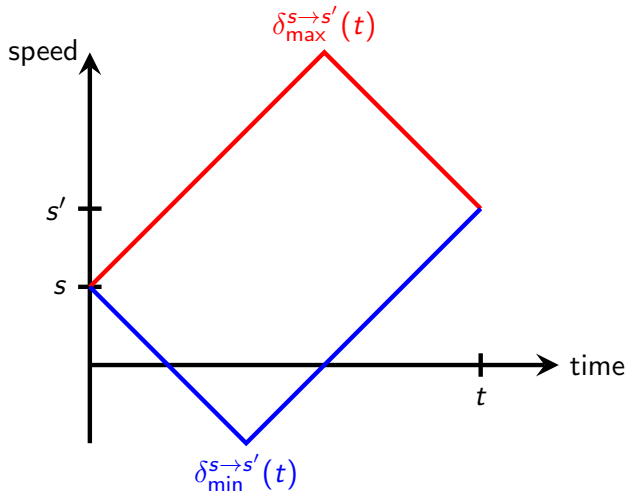
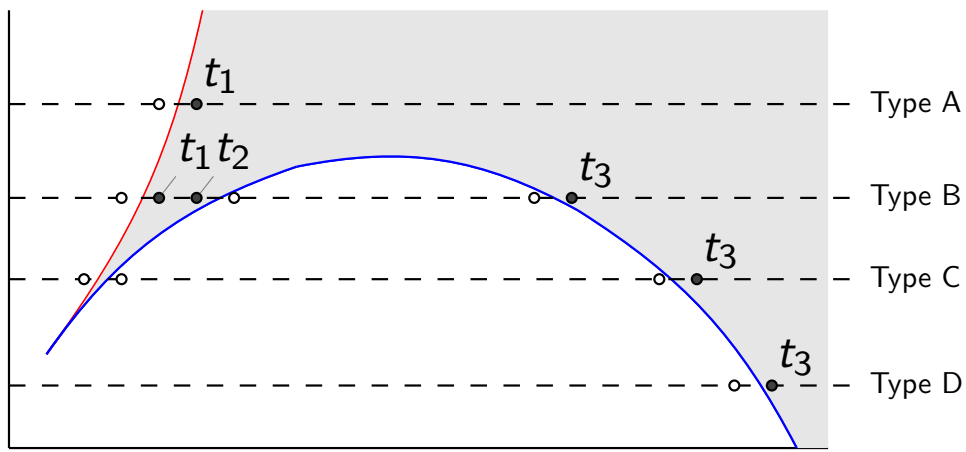
$\mathcal{I}(c, c')$ can always be described either as a single interval $\mathcal{I} = [t_1, \infty)$ or as two intervals $\mathcal{I} = [t_1, t_2] \cup [t_3, \infty)$.





Branching Cost d-dimensional

Given $c = ((x_1, x_2, \cdots, x_d), (s_1, s_2, \cdots, s_d))$ and
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$\rightarrow$ we retrieve the whole feasible interval as

$$\min \mathcal{I} = \bigcap_{i \in d} \mathcal{I}((x_i, s_i), (x'_i, s'_i))$$

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In $O(d \log d)$ time

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Problem: Branching Trajectory

In: Two configurations $c = (p, v)$ and $c' = (p', v')$ in $\mathbb{Z}^d \times \mathbb{Z}^d$

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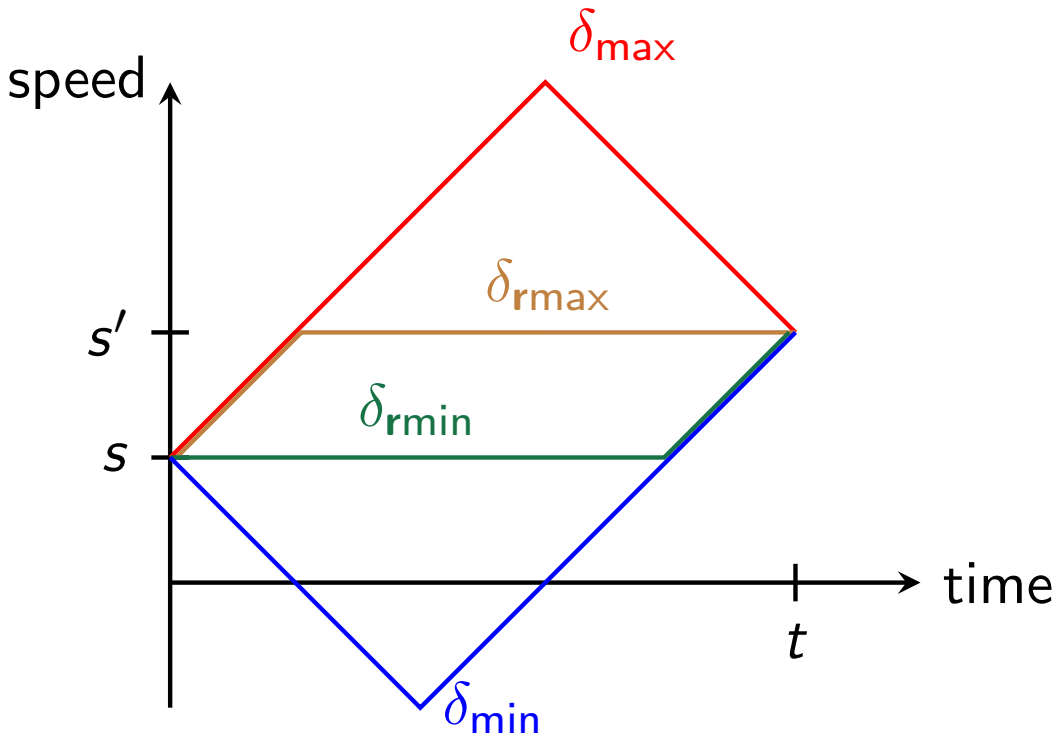
Pick a specific $t \in \mathcal{I}$. How to compute a shortest trajectory in t moves?

Again, we separate the computation in each dimension:

Branching Trajectory

Given t, s and s', we can calculate in constant time 4 quantities



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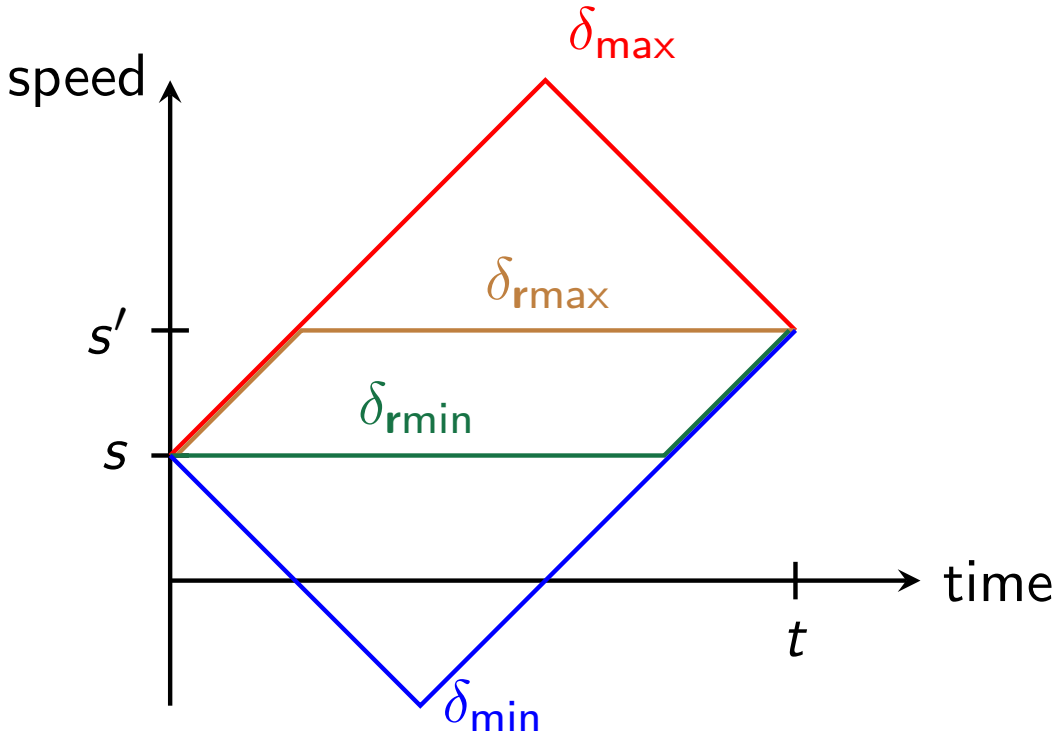


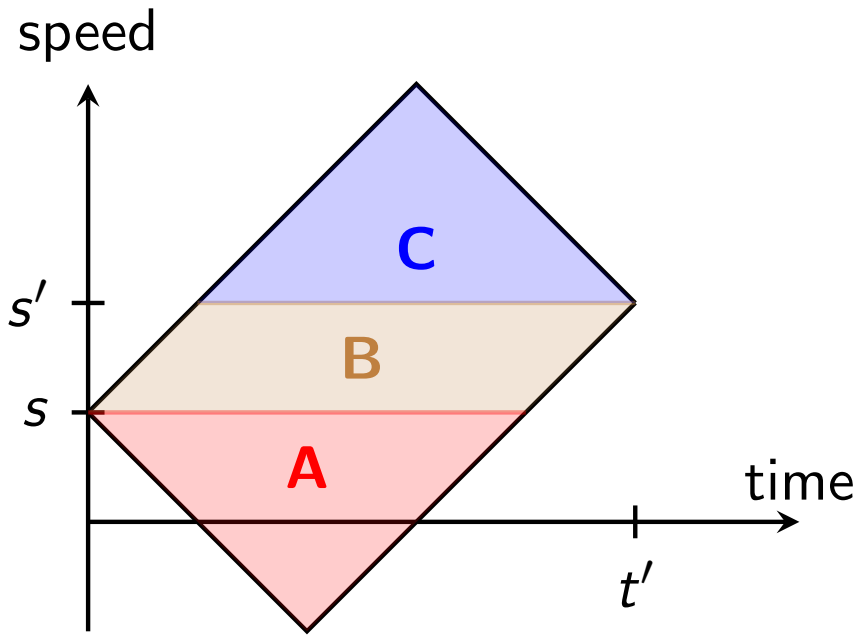
red trajectory: maximum distance covered from s and s' in t steps.

blue trajectory: minimum distance covered from s and s' in t steps.

brown and green: two additional reference distances.

Branching Trajectory

Since t is feasible, given the distance to cover $\delta = x' - x$, we can partition the *distances* space:



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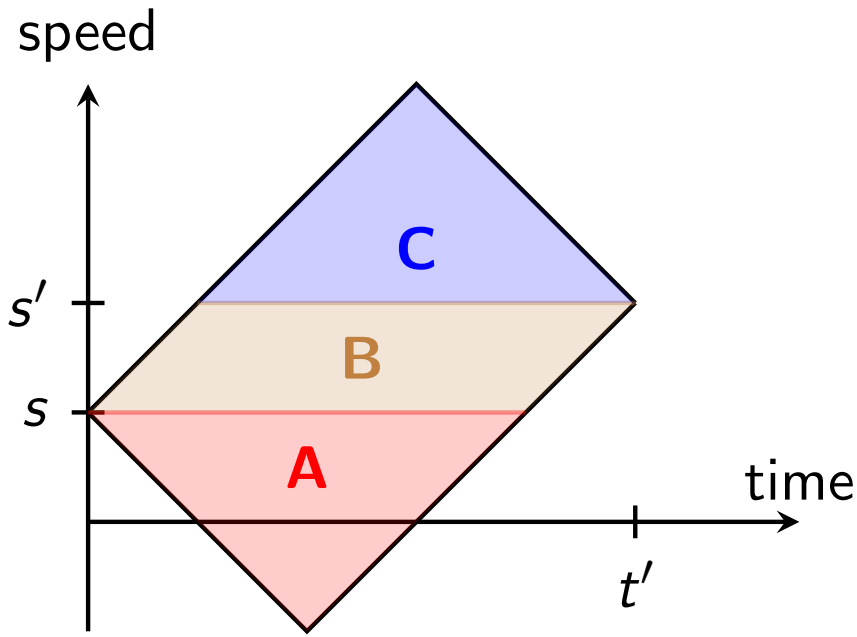

- $\delta_{\mathbf{r}\text{max}} < \delta \leq \delta_{\text{max}}$ (region **C**);
- $\delta_{\mathbf{r}\text{min}} < \delta \leq \delta_{\mathbf{r}\text{max}}$ (region **B**);
- $\delta_{\text{min}} \leq \delta \leq \delta_{\mathbf{r}\text{min}}$ (region **A**);

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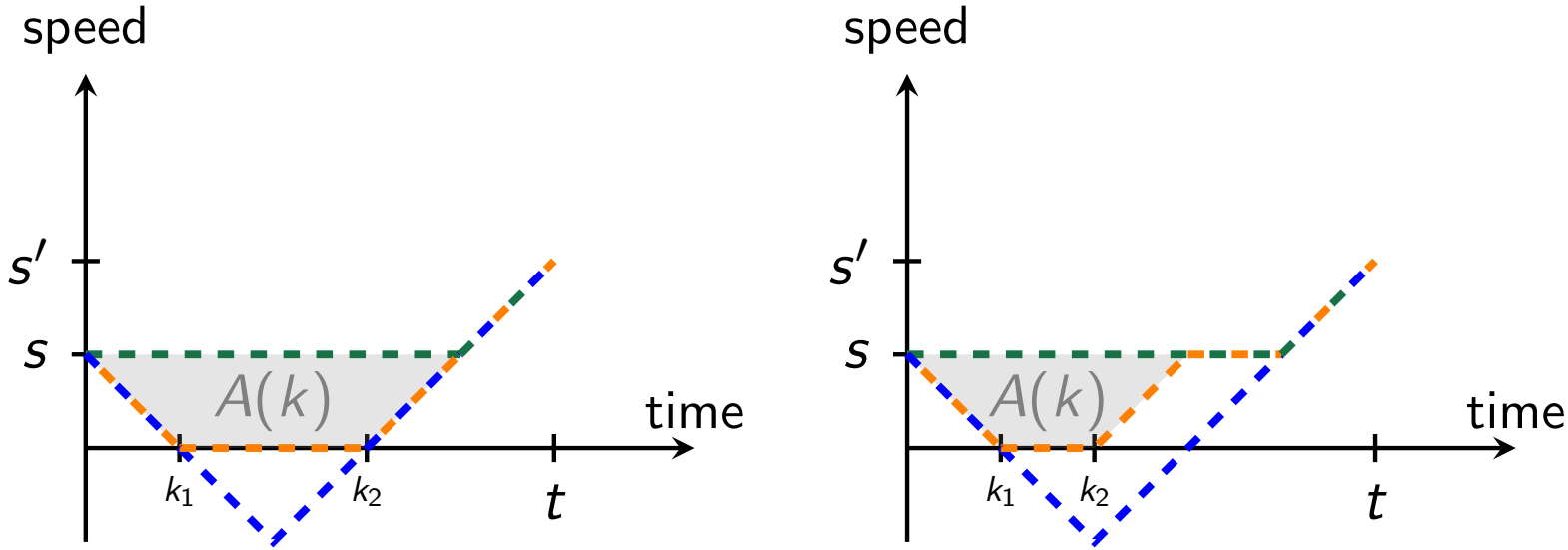
Suppose we are in region **A**: then a feasible δ trajectory has one of the following form



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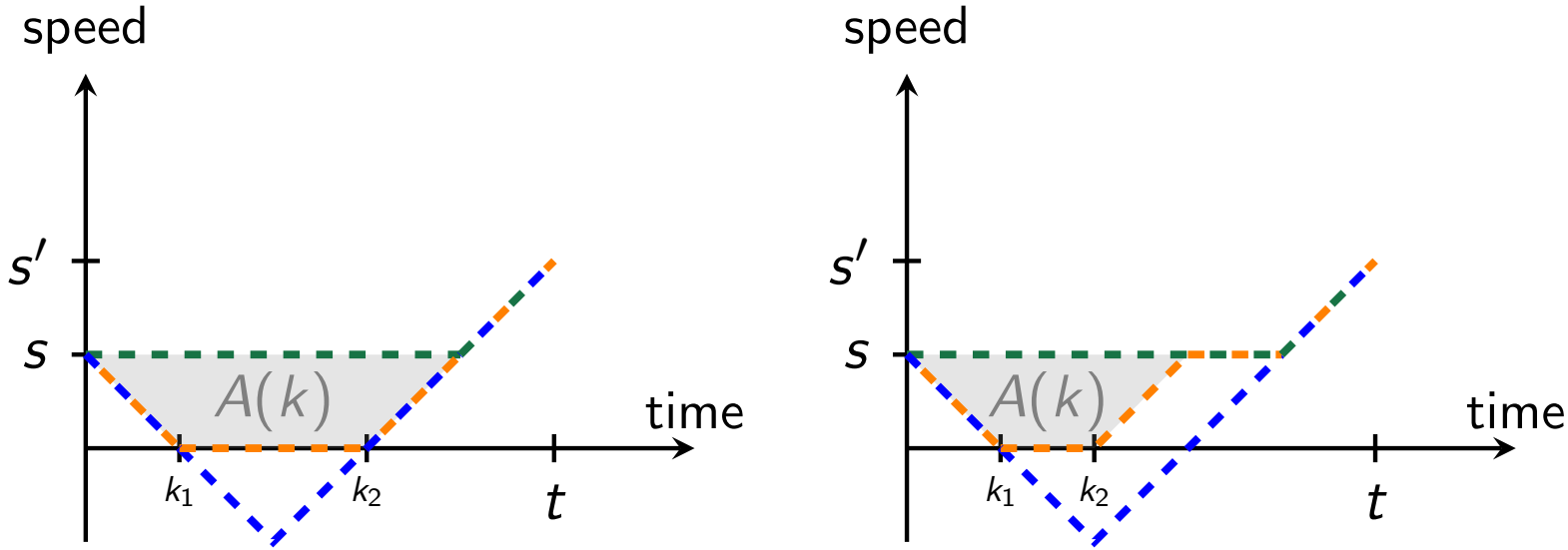

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Using checkpoints k_1 and k_2, we can describe a feasible trajectory in constant time and constant space.

$$(-, k_1)(0, k_2 - k_1)(+, \ell - k_2)$$

$$(-, k_1)(0, k_2 - k_1)(+, 1)(0, \ell - k_2 - k_1 - s' + s)(+, s' - s + k_1 - 1)$$

Multipoint Trajectory

Problem: Multipoint Trajectory

In: a sequence $x_0, x_1, \ldots, x_n$ of points in Euclidean space $\mathbb{Z}^d$
Out: a trajectory $\mathcal{T} = c_0, c_1, \ldots, c_\ell$ with $c_i = (p_i, v_i)$ such that $v_0 = v_\ell = \vec{0}$ and $\mathcal{T}$ visits the points $x_0, x_1, \ldots, x_n$ in this order. Furthermore, ℓ is minimum over all such trajectories.

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A configuration $c = (p, v)$ *visits* a point x if x lies on a segment between the points $p - v$ and p (the point must be traversed by the vector, not necessarily be located at its endpoint).

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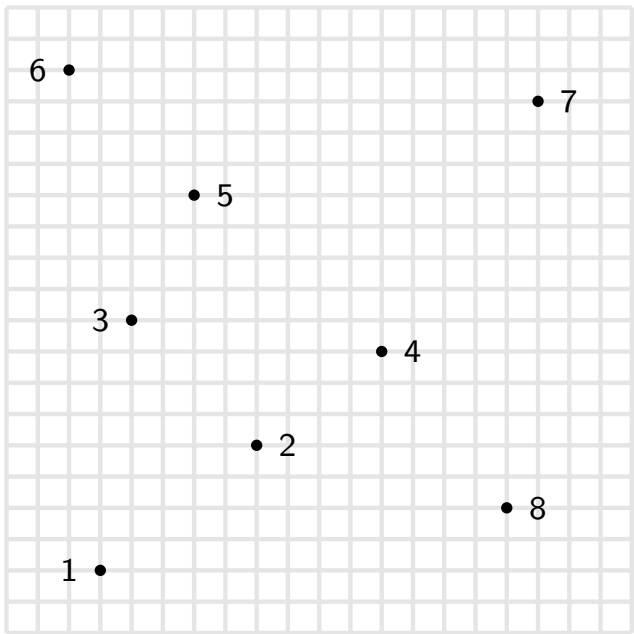

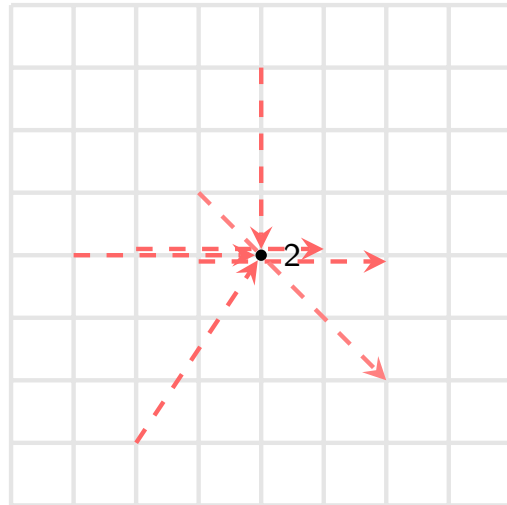

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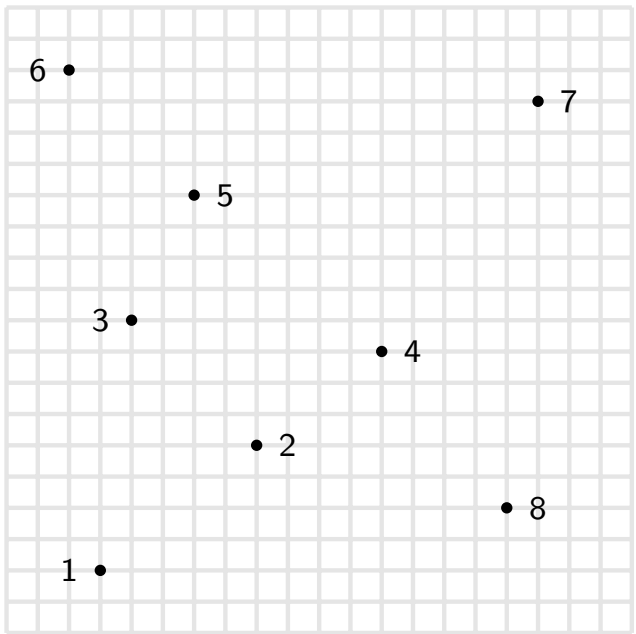

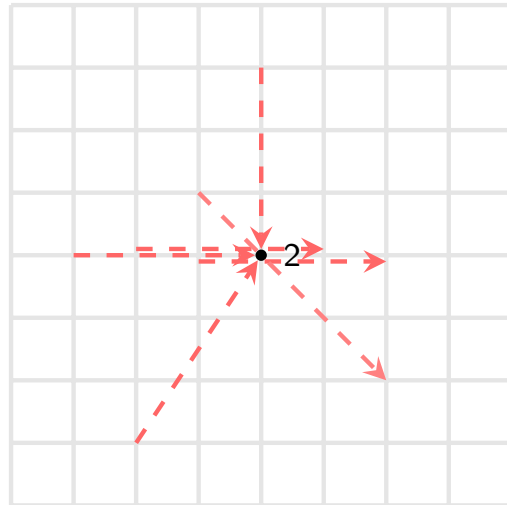

Dynamic programming algorithm, considering for every point all the configurations that can *visit* the point.

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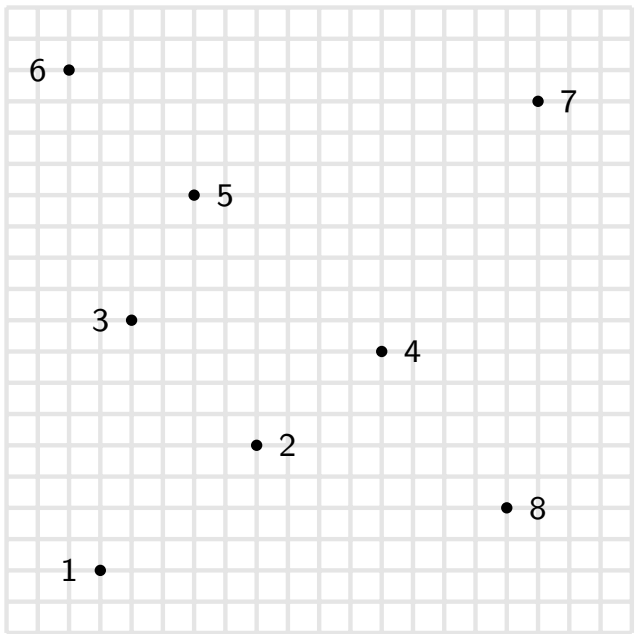

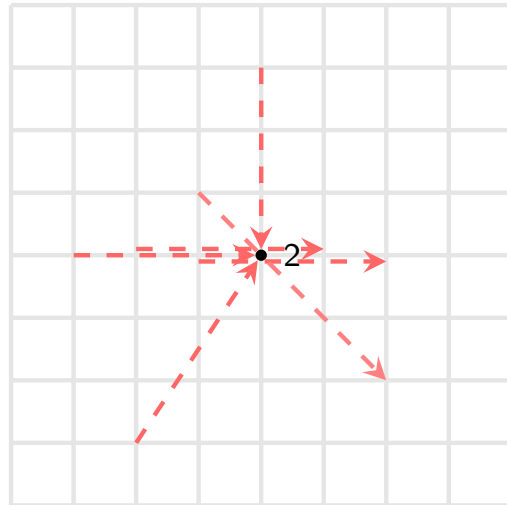

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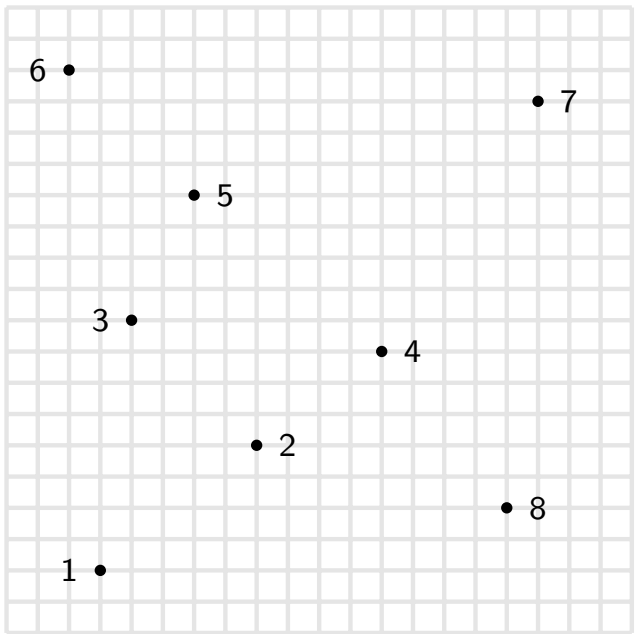

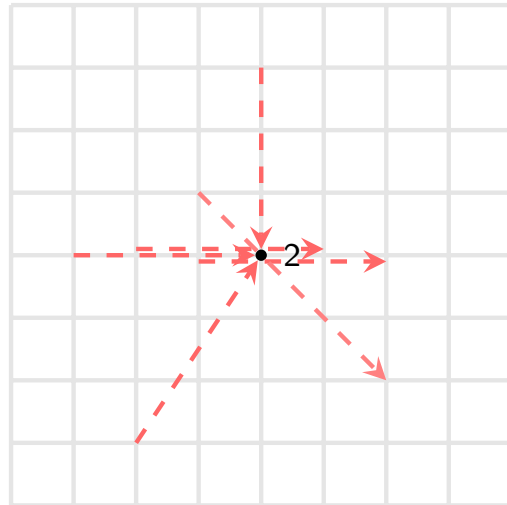

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The number of configurations grows with the speed: how to choose *locally maximum configurations speed* for every point?

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Given the following instance:

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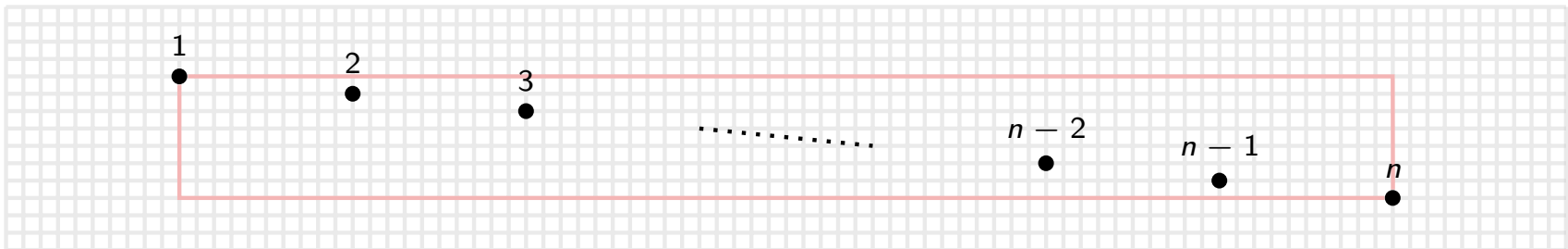

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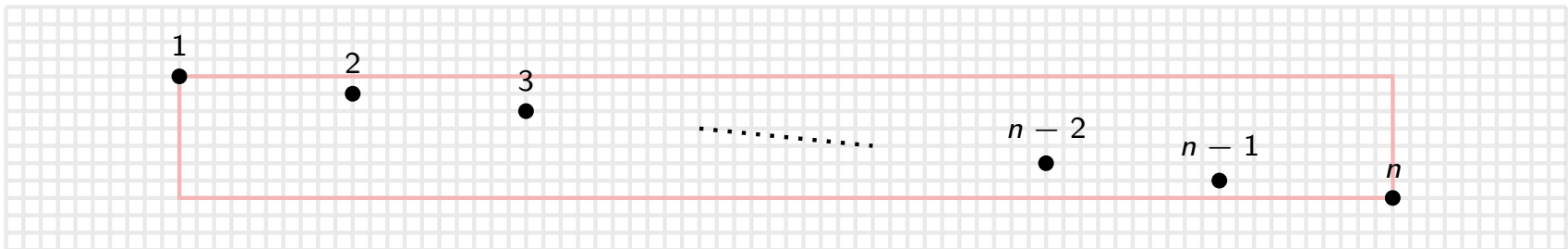

In red, *minimum bounding box* enclosing all the points.

Every point at distance δ (x-axis) and 1 (y-axis) from one another.

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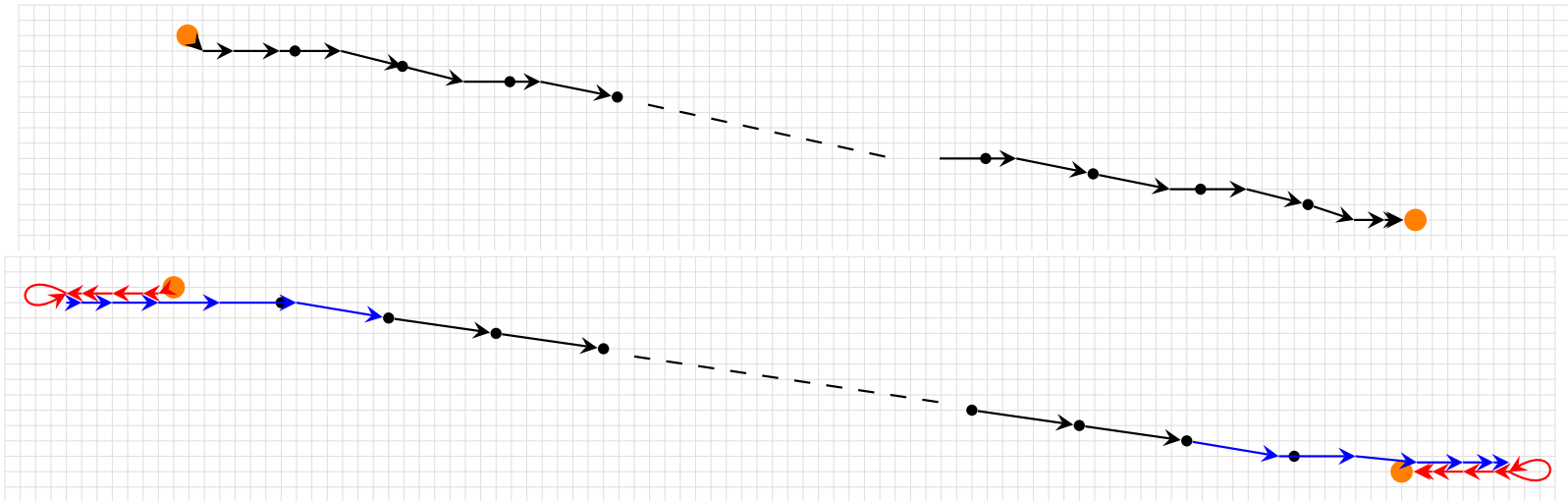

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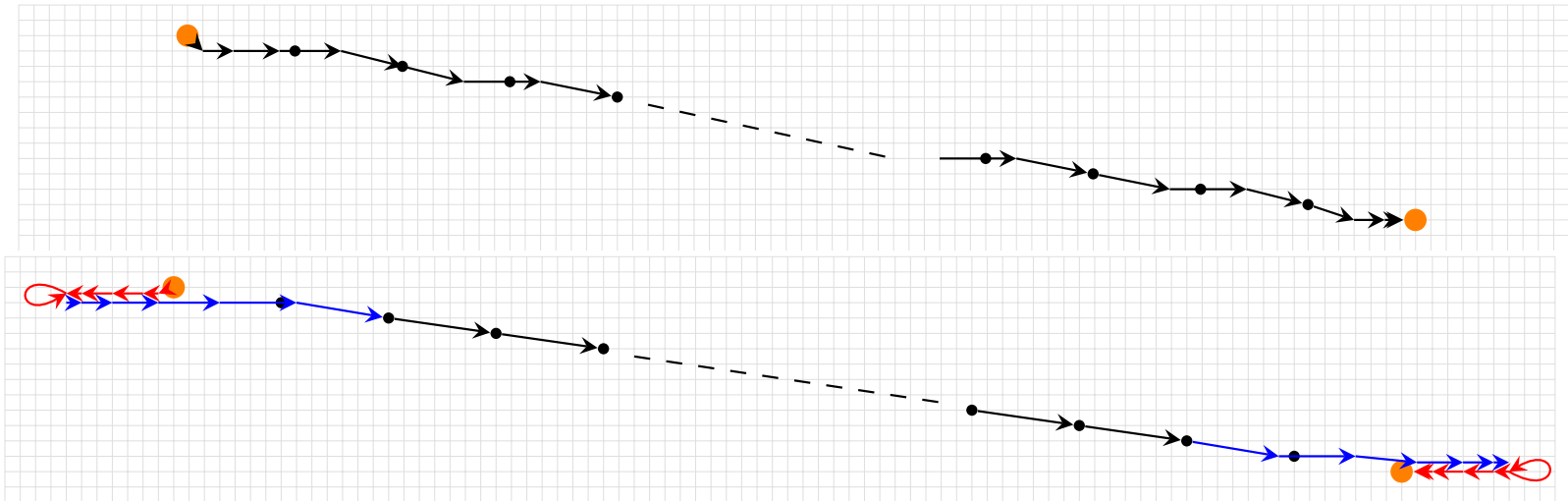

We distinguish two kind of trajectories:

- **NTU (Non-TUrning)** if it does not contain configurations with negative speed in the x dimension.
- **TU (TUrning)** if it allows for configurations with negative speed in the x dimension.

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We analytically derived the following result:

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Lemma 1 For all $\delta \geq 7$, any NTU trajectory cannot reach a x-speed of δ.

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Theorem 1 For any fixed $\delta \geq 7$ and constant $k \in \mathbb{Z}$, there exists a $n_0 = O(k^2\delta^2)$ such that $\forall n \geq n_0$, there exists an TU($k\delta$) with a better cost than an NTU.

Conjecture on the speed

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Experiments on random instances show the same phenomena.

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Conjecture 1 Let L be the maximum distance between two cities in any of the dimensions. Then, there exists an optimal trajectory that never exceeds speed $\sqrt{L}$ in any of the dimensions.

Experiments with/without conjecture

We implemented a Dynamic Programming (DP) algorithm in different settings:

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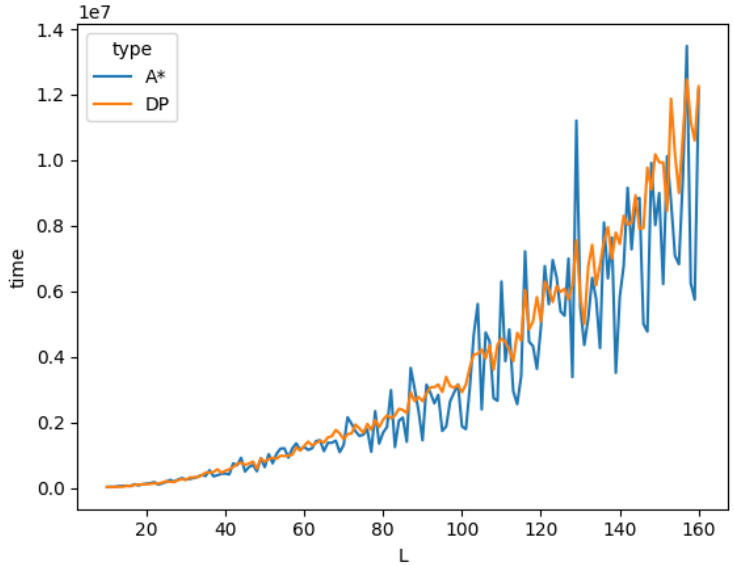
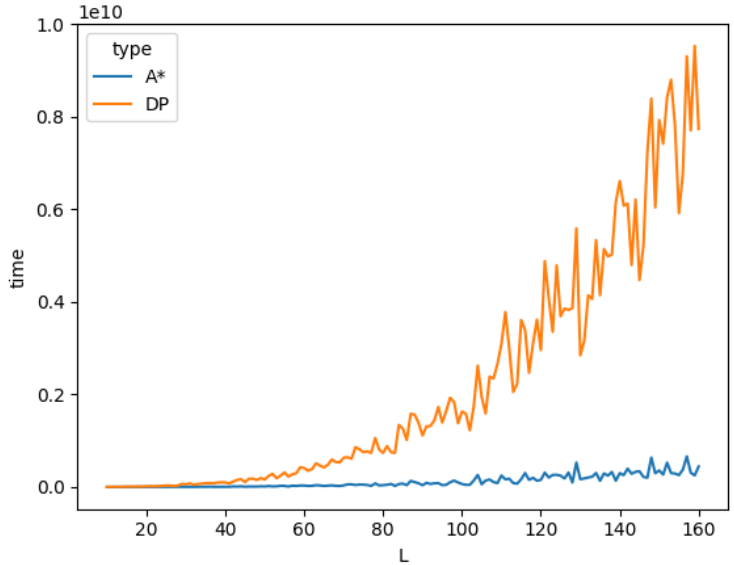
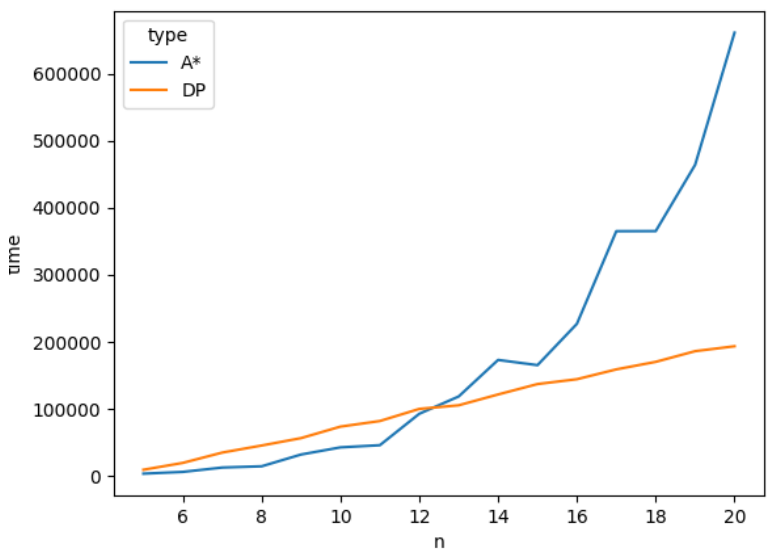
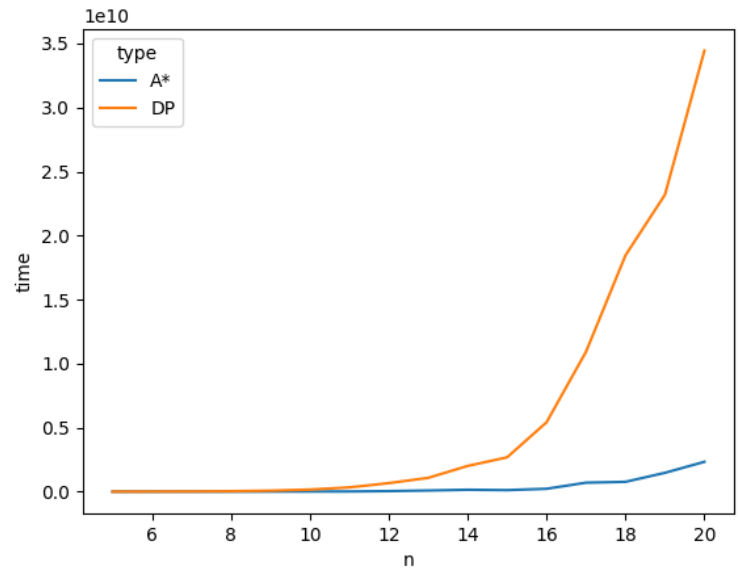
Comparing our DP with A^* we obtained

Experiments with/without conjecture









Open questions

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Are there families of instances where more (possibly locally) conservative bounds can be proven?

Thank you for the attention!

Questions?

Bibliography I

[1] M. Gardner, "Sim, chomp and race track-new games for intellect," *Scientific American*, vol. 228, no. 1, pp. 108–115, 1973.

[2] M. A. Bekos, T. Bruckdorfer, H. Förster, M. Kaufmann, S. Poschenrieder, and T. Stüber, "Algorithms and insights for racetrack," *Theoretical Computer Science*, 2018.